Assignment 7.

This homework is due *Thursday*, October 17.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much. Extra problems (if there are any) are due December 6.

1. Quick reminder

Function $f : E \to \mathbb{R} \cup \pm\infty$ is measurable if E is measurable and preimage of every open set is measurable. If E is measurable, to check that f is measurable it it enough to show that preimage of every interval of the form $(c, \infty]$ is measurable.

Sum and product of a.e. finite measurable function is measurable. Composition continuous $\circ$ measurable is measurable.

2. Exercises

(1) (3.1.1+) Suppose f and g are continuous on $[a, b]$. Show that if $f = g$ a.e. on $[a, b]$, then, in fact, $f = g$ on $[a, b]$. Is a similar assertion true if $[a, b]$ is replaced by a general measurable set E? By a general measurable set E of nonzero measure?

(2) (3.1.2) Let $E \subseteq R$ be measurable, D its measurable subset. We proved in class that a function $f : E \to \mathbb{R} \cup \pm\infty$ is measurable if and only if the restrictions $f|_D$, $f|_{E\setminus D}$ are measurable. Is the same true for continuous functions? That is, for $D \subseteq E \subseteq \mathbb{R}$, is it true that a function $f : E \to \mathbb{R}$ is continuous if and only if its restrictions $f|_D$ and $f|_{E\setminus D}$ are both continuous?

(3) (3.1.3) Suppose a function f has a measurable domain and is continuous except at a finite number of points. Is f necessarily measurable? (*Hint:* Use the theorem mentioned in problem 2.)

(4) ($\sim$3.1.4) It was proved in class that for a measurable function f and any $c \in \mathbb{R}$, the set $f^{-1}(c)$ is measurable. Prove that converse is not true: a function f with measurable domain such that $f^{-1}(c)$ is measurable for each $c \in \mathbb{R}$ may be not measurable. (*Hint:* Build a function whose restriction on a non-measurable set is strictly monotone. Tinker a bit.)

(5) (3.1.5) Suppose the function f is defined on a measurable set E and has the property that $\{x \in E \mid f(x) > c\}$ is measurable for each rational c. Is f necessarily measurable?